Lecture Notes: CLT Proof, Characteristic Functions, and Brownian Motion

December 16

1 CLT Proof (Continuation)

1.1 Review and Notation

Recall that $X_n \xrightarrow{d} X$ (convergence in distribution) if $E[g(X_n)] \to E[g(X)]$ for all $g \in C_b(\mathbb{R})$ (continuous bounded functions). By the Portmanteau Theorem, this is equivalent to convergence for all bounded uniformly continuous functions.

Definition 1 (Support and Compact Support)**.** *The support of a function is defined as* $Supp(g) = \overline{\{x : g(x) \neq 0\}}$. *We denote by* $C_c(\mathbb{R})$ *the space of continuous functions with compact support:*

$$C_c(\mathbb{R}) = \{g : \mathbb{R} \to \mathbb{R} \mid g \textit{ is continuous and has compact support}\}$$

Proposition 1. $X_n \xrightarrow{d} X$ *if* $E[g(X_n)] \to E[g(X)]$ *for all* $g \in C_c(\mathbb{R})$.

1.2 Proof of Proposition

Assume $E[g(X_n)] \to E[g(X)]$ for all $g \in C_c(\mathbb{R})$. We need to show that $F_{X_n}(t) \to F_X(t)$ at every continuity point t of F_X.

Fix $t \in \mathbb{R}$ such that F_X is continuous at t. Let $\epsilon > 0$. Since F_X is a CDF, $\lim_{x\to-\infty} F_X(x) = 0$ and $\lim_{x\to\infty} F_X(x) = 1$. Thus, we can find a large N such that:

$$F_X(-N) < \epsilon \quad \text{and} \quad F_X(N) > 1 - \epsilon \iff \mu_X(N, \infty) < \epsilon$$

We use this to approximate the indicator functions using partitions of unity.

Step 1: Lower Bound

We construct a function $g \in C_c(\mathbb{R})$ such that $\mathbf{1}_{[-N,t-\delta]} \leq g \leq \mathbf{1}_{(-\infty,t]}$. Then:

$$F_{X_n}(t) = \int \mathbf{1}_{(-\infty,t]} d\mu_{X_n} \geq \int g \, d\mu_{X_n}$$

Taking the liminf:

$$\liminf_{n\to\infty} F_{X_n}(t) \geq \lim_{n\to\infty} \int g \, d\mu_{X_n} = \int g \, d\mu_X$$
$$\geq \mu_X(-N, t-\delta) \geq F_X(t-\delta) - F_X(-N) \geq F_X(t) - 2\epsilon$$

Step 2: Upper Bound

We define a trapezoidal function $h \in C_c(\mathbb{R})$ to approximate the tail (t, ∞). Construct h such that $\mathbf{1}_{[t+\delta,N]} \leq h \leq \mathbf{1}_{[t,\infty)}$.

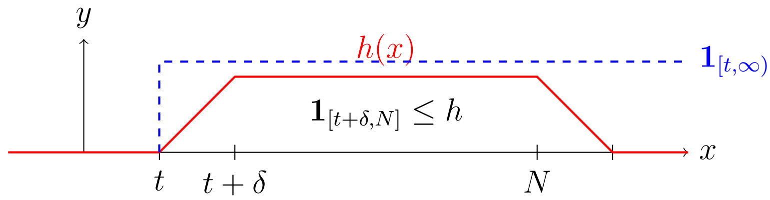

Figure 1: Approximation function h (red) relative to the indicator (blue).

Then:

$$F_{X_n}(t) = 1 - \mu_{X_n}(t,\infty) \leq 1 - \int h \, d\mu_{X_n}$$

Taking the limsup:

$$\limsup F_{X_n}(t) \leq 1 - \int h \, d\mu_X \leq 1 - \mu_X(t+\delta, N)$$

Using the bounds on N and continuity at t, we eventually get:

$$\limsup F_{X_n}(t) \leq F_X(t) + 2\epsilon$$

Combining Step 1 and 2 gives $\lim F_{X_n}(t) = F_X(t)$.

2 Lévy's Continuity Theorem

Theorem 1. *$X_n \xrightarrow{d} X$ if and only if $\varphi_{X_n}(t) \to \varphi_X(t)$ for all $t \in \mathbb{R}$, where $\varphi_X(t) = E[e^{itX}]$ is the characteristic function.*

2.1 Proof of ($\Rightarrow$) Direction

Assume $X_n \xrightarrow{d} X$. Since $g(x) = e^{itx}$ is a bounded continuous function for any fixed t, by the definition of convergence in distribution:

$$\varphi_{X_n}(t) = E[g(X_n)] \to E[g(X)] = \varphi_X(t)$$

2.2 Connection to Schwartz Space (Detailed Proof)

Let $g \in \mathcal{S}(\mathbb{R})$ (Schwartz space). We want to show $E[g(X_n)] \to E[g(X)]$. Using the Fourier Inversion Formula $g(x) = \int \hat{g}(t)e^{itx} \, dt$:

$$E[g(X_n)] = E\left[\int_{\mathbb{R}} \hat{g}(t)e^{itX_n} \, dt\right]$$

By Fubini's Theorem (applicable since quantities are bounded/integrable):

$$= \int_{\mathbb{R}} \hat{g}(t)E[e^{itX_n}] \, dt = \int_{\mathbb{R}} \hat{g}(t)\varphi_{X_n}(t) \, dt$$

Since $\varphi_{X_n} \to \varphi_X$ pointwise and is bounded by 1, by the Dominated Convergence Theorem:

$$\int \hat{g}\varphi_{X_n}\,dt \to \int \hat{g}\varphi_X\,dt = E[g(X)]$$

Density Argument ($\epsilon/3$ Proof)

We know that $\mathcal{S}(\mathbb{R})$ is dense in $C_c(\mathbb{R})$ (under the sup norm $||\cdot||_\infty$). Let $g \in C_c(\mathbb{R})$ and $\epsilon > 0$. There exists a function $g_\epsilon \in \mathcal{S}(\mathbb{R})$ such that $||g - g_\epsilon||_\infty < \epsilon$.

We know from the previous step that $E[g_\epsilon(X_n)] \to E[g_\epsilon(X)]$. Thus, for $n \geq n_0$, $|E[g_\epsilon(X_n)] - E[g_\epsilon(X)]| < \epsilon$. Also, since $||g - g_\epsilon||_\infty < \epsilon$:

$$||g(X_n) - g_\epsilon(X_n)||_\infty < \epsilon \implies E|g(X_n) - g_\epsilon(X_n)| < \epsilon$$

Similarly, $E|g(X) - g_\epsilon(X)| < \epsilon$.

Using the Triangle Inequality:

$$|E[g(X_n)] - E[g(X)]| \leq E|g(X_n) - g_\epsilon(X_n)| + |E[g_\epsilon(X_n)] - E[g_\epsilon(X)]| + E|g_\epsilon(X) - g(X)|$$

$$< \epsilon + \epsilon + \epsilon = 3\epsilon$$

Thus $E[g(X_n)] \to E[g(X)]$ for all $g \in C_c(\mathbb{R})$.

2.3 CLT (General Case)

Let $X_1, \dots, X_n$ be i.i.d. RVs. Assume $E[X_1] = \mu$ and $\mathrm{Var}(X_1) = \sigma^2 < \infty$ (where $\sigma > 0$). We define the standardized variables:

$$X_i' = \frac{X_i - \mu}{\sigma}$$

Then $E[X_i'] = 0$ and $\mathrm{Var}(X_i') = 1$. The standardized sum is:

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \frac{X_i - \mu}{\sigma}$$

This reduces to the standard version proved in the last lecture, so S_n (normalized) converges to $\mathcal{N}(0, 1)$.

3 Brownian Motion

3.1 Motivation and History

Observed by Robert Brown in 1828 (pollen/dust in liquid/air). Let $X(t)$ be the height above the ground of a dust mote at time t. Let $v(t) = X'(t)$ be the vertical velocity. By Newton's 2nd Law:

$$mv'(t) = F(t)$$

where $F(t)$ is the force generated by the collision of air molecules with the dust.

3.2 Reasonable Expectations

What is reasonable to expect from this motion?

1. **Independent Increments:** Given a moment in time $T > 0$, the increment in velocity in the future $[v(t+T) - v(t)]$ should be an independent copy of the previous increment $[v(T) - v(0)]$. This is because statistically, the force exerted by air molecules does not change with time (memoryless property).

2. **Normal Distribution:** The increments should be normally distributed because the motion is the net effect of the "bombardment" of millions of air molecules (CLT application).

3. **Linear Variance:** If $v(t+T) - v(0)$ is the sum of independent increments $v(t+T) - v(T)$ and $v(T) - v(0)$:

$$v(t+T) - v(0) = (v(t+T) - v(T)) + (v(T) - v(0))$$

$$\sim \mathcal{N}(\mu + \hat{\mu}, \sigma^2(t) + \sigma^2(T))$$

This implies $\sigma^2(T+t) = \sigma^2(T) + \sigma^2(t)$. The only continuous solution with $f(0) = 0$ is the linear function:

$$\sigma^2(T) = cT$$

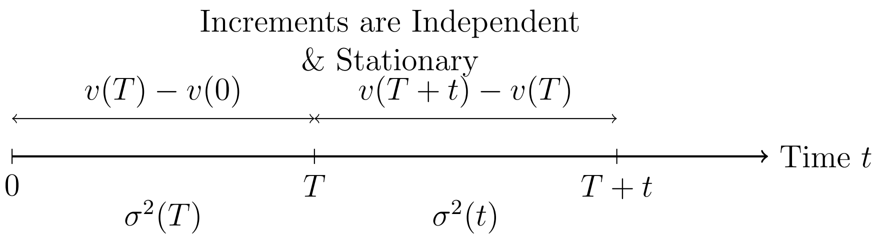

Figure 2: Visualizing the independent increments and additivity of variance over time.

3.3 Definition: 1D Brownian Motion

A Brownian Motion is a random process $\{B(t) : t \geq 0\}$ satisfying:

1. **Independent Increments:** For any $0 \leq t_1 < \cdots < t_n$, the increments $B(t_i) - B(t_{i-1})$ are independent.

2. **Gaussian Increments:** For any $t, h \geq 0$, $B(t+h) - B(t) \sim \mathcal{N}(0, h)$.

3. **Continuity:** Almost surely, $t \mapsto B(t)$ is continuous.

Theorem 2 (Wiener, 1923)**.** *Brownian Motion exists. There is a random process satisfying these conditions (Wiener Process).*