# HOMEWORK ASSIGNMENT FOR PART III

**Nota bene.** All exercises are due by 24 of January, 2026, sent by email at silviusk@puc-rio.br in one PDF file per student.

**Exercise 1.** Show that if $X$ is a random variable such that for some positive numbers $M, \epsilon, \delta$ we have

$$|X| \le M \text{ a.s.} \quad \text{and} \quad \mathbb{P}\big(\,|X| \ge \epsilon\big) \le \delta,$$

then

$$\mathbb{E}\,|X| \le \epsilon + M\,\delta.$$

**Exercise 2.** (the bounded convergence theorem in probability)

Let $X_1, X_2, \ldots$ be a sequence of random variables such that

$$\forall n \ge 1,\ |X_n| \le M \text{ a.s.} \quad \text{and} \quad X_n \to X \ \text{ in probability.}$$

Then

(a) $|X| \le M$ a.s.

(b) $\mathbb{E}X_n \to \mathbb{E}X$.

**Exercise 3.** (the bounded moments convergence theorem)

Let $X_1, X_2, \ldots$ be a sequence of random variables such that for some $p > 1$ and $M < \infty$,

$$\mathbb{E}\,|X_n|^p \le M \ \forall n \ge 1 \quad \text{and} \quad X_n \to X \ \text{ in probability.}$$

Then

$$\mathbb{E}X_n \to \mathbb{E}X.$$

**Exercise 4.** (approximation of the cumulative distribution function of a random variable)

Let $X_1, X_2, \ldots$ be i.i.d. copies of a real valued random variable $X$. Show that for every real number $t$ the following holds almost surely:

$$\frac{1}{n}\,\mathrm{card}\,\{1 \le i \le n \colon X_i \le t\} \to \mathbb{P}(X \le t) \ \text{ as } \ n \to \infty.$$

**Exercise 5.** Show that if $X_1, X_2, \ldots, X_n$ are i.i.d. copies of a real valued random variable $X$, and if $M$ is finite constant, then the truncations

$$X_1\,\mathbf{1}_{|X_1| \le M},\ X_2\,\mathbf{1}_{|X_2| \le M},\ \ldots,\ X_n\,\mathbf{1}_{|X_n| \le M}$$

are also independent and identically distributed random variables.

**Exercise 6.** Let $X_1, X_2, \ldots$ be i.i.d. real valued random variables. Prove that if $X_1 \ge 0$ a.s. and $\mathbb{E}X_1 = \infty$ then almost surely, $\dfrac{S_n}{n}$ diverges to infinity in probability, in the sense that for every $T < \infty$,

$$\mathbb{P}\big(\frac{S_n}{n} \ge T\big) \to 1 \ \text{ as } \ n \to \infty.$$

Hint: Truncate and use the weak LLN for the truncations. The truncation will have to be chosen carefully (you may need the monotone convergence theorem for that).

**Exercise 7.** (The law of large numbers for triangular arrays).

Let $X_{i,n}$, where $n \geq 1$ and $1 \leq i \leq n$ be a triangular array of random variables with the same mean $\mu$. Assume that each row $X_{1,n}, \ldots, X_{n,n}$ consists of independent random variables and let $S_n := X_{1,n} + \ldots + X_{n,n}$. Let $M$ be a finite constant. Prove the following:

(a) (weak LLN) If $\mathbb{E}\,|X_{i,n}|^2 \leq M$ for all indices, then $\frac{S_n}{n} \to \mu$ in probability.
(b) (strong LLN) If $\mathbb{E}\,|X_{i,n}|^4 \leq M$ for all indices, then that $\frac{S_n}{n} \to \mu$ a.s.

**Exercise 8.** Let $T$ be a stopping time adapted to a given filtration.

(a) Prove that if $M$ is a martingale (relative to the given filtration), then the stopped process $M^T$ is also a martingale.
(b) Prove that if $A$ is a predictable process (relative to the given filtration), then te stopped process $A^T$ is also predictable.

**Exercise 9.** Let $C$ be a predictable and bounded process. Prove that if $M$ is a martingale, then its martingale transform by $C$, namely $M * C$, is a martingale too, and it is null at zero.

Let $X$ be a scalar random variable and let $\mu_X$ be its probability distribution. Compute:

(a) Then mean $\mu$;
(b) The standard deviation $\sigma$;
(c) The characteristic function $\varphi_X(t) := \mathbb{E}\, e^{itX}$

for the following examples.

**Exercise 10.** The Bernoulli r.v. with values 1 and $-1$ with equal probabilities $\frac{1}{2}$.

**Exercise 11.** The standard normal distribution $N(0,1)$.

If you do the calculation correctly, you should get $\varphi(t) = e^{-t^2/2}$.

**Exercise 12.** The uniform distribution of the interval $(0,1)$.

The next problem will tell us that under appropriate assumptions, derivates and integrals may be interchanged (just like limits and integrals may be interchanged). It was needed in the proof of the central limit theorem.

**Exercise 13.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $I \subset \mathbb{R}$ be an interval. Given any absolutely integrable function $f\colon I \times \Omega \to \mathbb{C}$, define

$$\varphi(t) := \int_\Omega f(t,\omega) d\mu(\omega).$$

We assume that for every $\omega \in \Omega$, the function $I \ni t \mapsto f(t,\omega) \in \mathbb{C}$ is differentiable at every point $t \in I$ and that its derivative satisfies

$$\left| \frac{d}{dt}\, f(t,\omega) \right| \leq g(\omega) \quad \text{for all } t \in I,$$

where $g \in L^1(\Omega, \mu)$.

Prove that $\varphi$ is differentiable at all points $t \in I$ and

$$\frac{d}{dt}\,\varphi(t) = \int_\Omega \frac{d}{dt}\, f(t,\omega)\, d\mu(\omega).$$

*Hint:* Interpret the derivative as a limit and use dominated convergence.

**Exercise 14.** Prove that if $X\colon \Omega \to \mathbb{R}$ is a random variable with $\mathbb{E}\,|X| < \infty$, then its characteristic function $\varphi_X(t)$ is differentiable everywhere and

$$\varphi_X'(t) = i\,\mathbb{E}\,(Xe^{itX}).$$

*Hint:* Apply the previous exercise to the function $f\colon \mathbb{R}\times\Omega \to \mathbb{C}$, defined by $f(t,\omega) := e^{i\,t\,X(\omega)}$.

**Exercise 15.** Let $S$ be a metric space and let $\mu$ be a Borel probability measure on $S$. Prove that if $g\colon S \to \mathbb{R}$ is a continuous function, then the set

$$\Big\{ t \in \mathbb{R} \colon \mu\{g = t\} > 0 \Big\}$$

is at most countable.

**Exercise 16.** Let $(X_n)_n$ be a sequence of Gaussian random vectors. Assume that $\mathbb{E}X_n \to b$ and $\mathrm{cov}(X_n) \to C$ as $n \to \infty$. Prove that if $X_n \to X$ a.s. then $X$ is a Gaussian random vector with $\mathbb{E}X = b$ and $\mathrm{cov}(X) = C$.

**Exercise 17.** In the construction of the Brownian motion done in class, when we established the independent increments property for dyadic times $d \in \mathcal{D}$, we actually only treated the particular cases of times in $\mathcal{D}_1$ and $\mathcal{D}_2$. Prove the general case by induction.

**Exercise 18.** Let $S$ be a metric space, endowed with the Borel $\sigma$-algebra. By definition, a sequence $(X_n)_n$ of random values with values in $S$ converges in distribution to another $S$-valued random variable $X$ if $\mu_{X_n} \to \mu_X$ weakly as $n \to \infty$. Recall that $\mu_X$ is the probability measure on $S$ given by $\mu_X(A) = \mathbb{P}(X \in A)$.

Prove that if $X_n \implies X$ and $F\colon S \to \mathbb{R}$ is continuous, then $F(X_n) \implies F(X)$.

**Exercise 19.** Let $(S_n)_{n\ge 0}$ be a random walk with $S_0 = 0$ and independent, identically distributed increments with mean 0 and variance 1. Define the process

$$M_n := \max\{S_0, S_1, \dots, S_n\}.$$

Use the functional CLT (Donsker's invariance principle) to prove that for every $\lambda > 0$, we have

$$\lim_{n\to\infty} \mathbb{P}(M_n \ge \lambda\sqrt{n}) = \mathbb{P}(\sup_{t\in[0,1]} B(t) \ \ge \lambda).$$

**Observation.** It turns out that the limit above has a closed form formula, namely

$$\mathbb{P}(\sup_{t\in[0,1]} B(t) \ \ge \lambda) = 2\,\mathbb{P}(B(t) \ge \lambda) = 2\int_\lambda^\infty e^{-x^2/2}\,\frac{dx}{\sqrt{2\pi}}\,,$$

but this is a property of the Brownian motion that we have not studied (it is called the reflection principle).

**Exercise 20.** Let $(M,d)$ and $(N,d)$ be two metric spaces, let $V \subset M$ be a subset (say a ball) and let $f_n\colon M \to N$, $n \ge 1$, be a sequence of functions. Assume the following:

(i) The sequence $\{f_n\}_n$ converges uniformly on $V$ to a function $f$ at an exponential rate, i.e. for some $c > 0$ we have

$$d(f_n(a), f(a)) \le e^{-c\,n} \quad \text{for all } a \in V \quad \text{and for all } n \ge 1\,.$$

(ii) There is $C > 0$ such that for all $a, b \in V$ and for all $n \ge 1$,

$$\text{if} \quad d(a,b) \le e^{-C\,n} \quad \text{then} \quad d(f_n(a), f_n(b)) \le e^{-c\,n}\,.$$

Then for all $x, y \in V$ we have

$$d(f(x), f(y)) \le 3\,e^c\,d(x,y)^{\frac{c}{C}}\,,$$

that is, $f$ is Hölder continuous on $V$ with Hölder exponent $\alpha = \frac{c}{C}$.